ISRAEL JOURNAL OF MATHEMATICS, Vol. 52, Nos. 1-2, 1985

INVARIANTS FOR ω-CATEGORICAL, ω-STABLE THEORIES

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ABSTRACT

In this paper we give a complete solution to the classification problem for ω-categorical, ω-stable theories. More explicitly, suppose T is ω-categorical, ω-stable with fewer than the maximum number of models in some uncountable power. We associate with each model M of T a "simple" invariant $\mathcal{I}(M)$, not unlike a vector of dimensions, such that $\mathcal{I}(M)=\mathcal{I}(N)$ if and only if $M \cong N$. The spectrum function, $I(-,T)$, for a first-order theory T is such that for all infinite cardinals λ, $I(\lambda,T)$ is the number of nonisomorphic models of T of cardinality λ. As an application of our "structure theorem" we determine the possible spectrum functions for ω-categorical, ω-stable theories.

Introduction

It is generally accepted that Shelah has solved the classification problem (described above) for countable first-order theories. (See [14] and [2, I.1] for a complete description of the problem and a statement of the results.) However, Shelah's structure theorem (assignment of invariants) is imprecise in that there is not a 1-1 correspondence between models and invariants. This is generally overlooked since, at least for ω-stable theories, it is still possible to determine the spectrum function of the theory (see [12]). The deficiency, however, remains. We will prove an exact structure theorem for ω-categorical, ω-stable theories by using a different assignment of invariants.

Suppose T is ω-categorical, ω-stable with $I(\lambda,T)<2^{\lambda}$ for some $\lambda>\omega$. Shelah associates with $M\models T$ a tree of countable submodels, called a *representation* of M, over which M is prime. It has not been shown that all representations of M are isomorphic, but only "quasi-isomorphic". It is at this point that the imprecision in Shelah's structure theorem occurs. Here we associate with M a

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Received June 18, 1984 and in revised form April 28, 1985

tree of *elements* of M^{eq}, called a *coordinate tree* of M. It is not difficult to show that the coordinate trees of M are all isomorphic. As with representations M is prime over its coordinate tree, but this is much harder to prove than in the representation case. It is at this point where the concept of a "filtration" arises, and where we make essential use of the Coordinatization Theorem ([4, 4.1]).

As a test of our structure theorem we prove

THEOREM. *Suppose T is ω-categorical, ω-stable. Then $I(\_\,, T)$ is one of*

(a) $I(\lambda, T) = 2^{\lambda}$ *for all* $\lambda > \aleph_0$;

(b) *for* $\alpha < \omega$, $I(\aleph_\alpha, T)$ *is a finite number explained in* 3.3,
for $\alpha \geqq \omega$, $I(\aleph_\alpha, T) = |\alpha|$;

(c) $I(\lambda, T) = 1$ *for all* $\lambda \geqq \aleph_0$;

(d) $I(\aleph_\alpha, T) = \beth_{d-2}((|\alpha| + \aleph_0)^{|\alpha|})$ *for* $\alpha > 0$ *and some* $d \in \omega$.

Case (b) occurs when T is non-multidimensional. The number d in (d) is what Shelah calls the depth of T.

§1. Preliminaries and notation

We assume a basic knowledge of stability theory as found in [10], [11] and [13, III]. For the most part our notation follows [10]. The type of A over B is denoted $\mathrm{t}(A/B)$, even when A is infinite. We write $A \downarrow_C B$ or $(A \downarrow B/C)$ for $\mathrm{t}(A/B \cup C)$ does not fork over C. If p is stationary and there is a type $q \in S(B)$ parallel to p we write $q = p \mid B$. For $\mathrm{t}(a/A) = \mathrm{t}(b/A)$ we write $a \equiv b(A)$; $a \equiv^s b(A)$ denotes $\mathrm{stp}(a/A) = \mathrm{stp}(b/A)$. For $p \in S(A)$ we say q is a *strong type of* p if there is an a realizing p such that $q = \mathrm{stp}(a/A)$. When convenient we write AB for $A \cup B$; Aa for $A \cup \{a\}$. When a type p is orthogonal to the set A we write $p \dashv A$.

We assume every set is a subset of a large saturated model $\mathfrak{C}$ called the monster model. Every model is considered to be an elementary submodel of $\mathfrak{C}$. See [13, p. 7] for a complete discussion. In this paper we will often work in $\mathfrak{C}^{\mathrm{eq}}$, as discussed in [10, §A] and [13, III, §6]. If M^* is the restriction of M^{eq} to finitely many new sorts we call M^* an *extension by definitions of M.*

For p a type and A a set we let $p(A) = \{b \in A : b$ realizes $p\}$. We say $H \subset M$ is A*-definable* if there is a formula φ over A such that $H = \varphi(M)$; if $A = \emptyset$ we say 0-definable. A and B are *conjugate over* C if $A \equiv B\ (C)$. For p a type over A, q a type over B we say p and q are *conjugate over* C if there is an automorphism α fixing C such that $\alpha(p) = q$. If $A = \varphi(M)$ and φ is a complete formula A is called an *atom*. Morley rank is denoted $\mathrm{rk}(-)$.

We abbreviate ω-categorical, ω-stable by ω-c.s. What we say from here on only applies to ω-c.s. theories.

We assume some familiarity with the results in [4], although in general, not the proofs. A rank 1 atom A over B is called *reduced* if $a \in A$ implies $\mathrm{acl}(aB) \cap A = \{a\}$. For any rank 1 atom A there is an E such that A/E is reduced. We say a reduced rank 1 atom A over B is *trivial* if whenever $a \in A$, $X \subset A$ and $a \in \mathrm{acl}(XB)$, $a \in X$.

We refer the reader to [4] for the definition of local modularity and modularity (as properties of strongly minimal sets). Independently, Cherlin and Zil'ber proved that every strongly minimal set in an ω-c.s. theory is locally modular [4], [15]. This is the result which yields the theorems in [4]. Model-theoretically the most important property of locally modular sets is the following. Note that if $\mathrm{rk}(a/A) = 1$ then $a \not\downarrow_A b \Leftrightarrow a \in \mathrm{acl}(Ab)$.

LEMMA 1.1. *Suppose H and G are A-definable, non-orthogonal strongly minimal atoms.*

(i) *For all $a \in H$, $b \in G$ there are $a' \in H$, $b' \in G$ such that $aa' \not\downarrow_A bb'$.*

(ii) *If both H and G are modular there are $a \in H$, $b \in G$ such that $a \not\downarrow_A b$.*

The major technical lemma in [4] is

LEMMA 1.2. (Coordinatization theorem) *Suppose P is a B-definable atom in M. Then in some extension by definitions M^* there is a B-definable atom A such that*

(i) *A is reduced and has rank* 1,

(ii) *for all $a \in P$* $\mathrm{acl}(aB) \cap A \neq \emptyset$.

LEMMA 1.3. *If $p \in S_1(M)$ there is a singleton $a \in M$ such that p does not fork over a. Furthermore, every infinite set of indiscernibles I is based on any $e \in I$.*

PROOF. This follows from the proof of the finiteness of the fundamental order in [4].

LEMMA 1.4. *Suppose $A \subset \mathfrak{C}^{eq}$ is finite. Then there is no infinite set of types $\mathcal{P}$ such that* (*) *each element of $\mathcal{P}$ is non-orthogonal to A and the elements of $\mathcal{P}$ are pairwise orthogonal.*

PROOF. Suppose there is such a $\mathcal{P}$. From simple facts about $\mathfrak{C}^{eq}$ we know that for each $p \in \mathcal{P}$ there is a regular $q_p \in S_1(\mathfrak{C})$ such that $p \not\perp q_p$ and $q_p \not\perp A$ (the variable in q_p ranges over the original universe). Let $\mathcal{P}' = \{q_p : p \in \mathcal{P}\}$. $\mathcal{P}'$ also satisfies (*). By 1.3 there is for each $q \in \mathcal{P}'$ an a_q such that q is definable over a_q and $\ell h(a_q) = 2$. By the ω-categoricity and ω-stability there are $a_q \neq a_p$ such that

$a_q \equiv^s a_p$ (A) and p is conjugate to q over A. Since $q \not\perp A$ [10, C.6] would imply $p \not\perp q$ if we had $a_q \downarrow_A a_p$. However it's not hard to show this additional condition is not needed when p and q are regular (see, e.g., the proof of 3.4 in [3]). Thus, $p \not\perp q$, contradicting our assumptions on $\mathcal{P}'$ to prove the lemma.

In [4] it is proved that the rank of T is finite. Since U-rank and Morley rank are the same on ω-c.s. theories ([7] or [9]) we can restate 5.8 of [8]:

LEMMA 1.5. $\mathrm{rk}(ab/A) = \mathrm{rk}(a/A \cup b) + \mathrm{rk}(b/A)$.

We can now state and prove the main technical lemma. In arbitrary ω-stable theories such a result is only true when A is a model. It's this lemma which allows us to obtain a tree of elements over which M is prime, rather than just a tree of submodels. In the proof we use Shelah's notion of *canonical base* (see [13, III, 6.10]). For ease of reference we restate [13, III, 6.10(5)].

LEMMA 1.6. *For any stationary p, if p does not fork over B then* $\mathrm{Cb}(p) \subset \mathrm{acl}(B)$.

LEMMA 1.7. *Suppose $\varphi(x,a)$ is a strongly minimal atom which is non-orthogonal to the finite set A. Then there is a rank* 1 *atom ψ over A such that $\varphi \not\perp \psi$.*

PROOF. By adding constants to the language we may assume $A = \varnothing$. If $\varphi(x,a)$ does not fork over $\varnothing$ we can take its restriction to $\varnothing$ as ψ. Thus, we assume $\varphi(x,a)$ forks over $\varnothing$. We know φ is locally modular. By replacing φ by its non-forking extension over some realization of it we may assume that it is modular.

Find an a' satisfying

(1) $$a' \equiv^s a \quad \text{and} \quad a' \downarrow a.$$

Since $\varphi(x,a) \not\perp \varnothing$, [10, C.6] implies that $\varphi(x,a) \not\perp \varphi(x,a')$. Applying 1.1(ii) (with $A = aa'$) we find e satisfying $\varphi(x,a)$ and e' satisfying $\varphi(x,a')$ such that

(2) $$e \downarrow_a a',\ e' \downarrow_{a'} a \quad \text{and} \quad e \not\downarrow_{aa'} e'.$$

Notice that by (1), (2) and the transitivity of non-forking we have

(3) $$ea \downarrow a' \quad \text{and} \quad e'a' \downarrow a.$$

Let I be an infinite Morley sequence in $\mathrm{stp}(e'a'/ea)$, $p = \mathrm{Av}(I/\mathfrak{C})$. Let $c \in \mathrm{Cb}(p) \setminus \mathrm{acl}(\varnothing)$ (there is such a c since $\varphi(x,a)$ forks over $\varnothing$). By the definition of a Morley sequence we know p does not fork over ea. By 1.6 and the fact that $c \notin \mathrm{acl}(\varnothing)$ we have

(4) $$c \in \mathrm{acl}(ea) \quad \text{and} \quad c \not\downarrow ea.$$

By 1.3, p does not fork over $e'a'$, so $c \in \mathrm{acl}(e'a')$. Thus (3) yields $ce'a' \downarrow a$, in particular,

$$(5) \qquad c \downarrow a.$$

By (4) and the transitivity of non-forking $e \not\downarrow_a c$. Since $\mathrm{rk}(e/a) = 1$ we have $e \in \mathrm{acl}(ca)\backslash\mathrm{acl}(a)$. By (4) and (5) $c \in \mathrm{acl}(ea)\backslash\mathrm{acl}(a)$. We conclude that $\mathrm{rk}(c/a) = 1$. By (5) $\mathrm{rk}(c) = 1$ and we may take ψ to be some complete formula in $t(c)$. This proves the lemma.

COROLLARY 1.8. *Let $\varphi(x, a)$ be a rank 1 atom non-orthogonal to A. Then there is a rank 1 formula ψ over A such that every strong type of $\varphi(x, a)$ is non-orthogonal to ψ.*

PROOF. φ is non-orthogonal to A when one of its strong types is. Since φ is an atom all of its strong types are conjugate over A, hence all are non-orthogonal to A. Now take as ψ the disjunction of the rank 1 sets obtained by applying 1.7 to each strong type of φ.

We will make use of the following results of Shelah. They follow easily from 2.2 and 2.3 of [5] and the definition of DOP. For brevity we say *T has many models* if in every $\lambda > \omega$ T has 2^λ non-isomorphic models. If T doesn't have many models we say it has *few models*. Note that $\mathrm{rk}(p) = 1 \Rightarrow$ the weight of p is 1.

LEMMA 1.9. (i) *Suppose* $\mathrm{t}(a/A)$ *has rank* 1 *and is non-trivial*; $p \in S(aA)$ *has rank* 1 *and is orthogonal to A. Then T has many models.*

(ii) *Suppose* $\mathrm{rk}(a/A) = \mathrm{rk}(b/A) = 1$ *and* $a \downarrow_A b$; $p \in S(abA)$ *has rank* 1 *and* $p \dashv aA$ *and* $p \dashv bA$. *Then T has many models.*

Let p be a reduced rank 1 type (over $\varnothing$, for simplicity). A word is in order concerning what a basis for p looks like, and the relationship between p and its strong types. First suppose that the strongly minimal components (i.e., the strong types) of p are modular (if one is modular they all are). Let q_0, q_1 be strong types of p (over $\mathrm{acl}(\varnothing)$ in M^{eq}). If $q_0 \not\perp q_1$ then, by 1.1, $q_0 \not\perp^a q_1$, so there are a_i realizing q_i $(i = 0, 1)$ with $a_0 \in \mathrm{acl}(a_1)$. This contradicts that p is reduced. Thus, if p is modular, its strong types are pairwise orthogonal. A basis for p looks like $B = B_0 \cup \cdots \cup B_m$, where the B_i's are bases for the strong types of p in M. If p is trivial then $B = p(M)$. Now suppose the strong types of p are non-modular. In this case the strong types may not be pairwise orthogonal. There is a basis $B = B_0 \cup \cdots \cup B_m \cup A$, where B_i is a basis for the strong type q_i, every strong type of p is non-orthogonal to some q_i, and A is a minimal finite set such that $\mathrm{acl}(B)$ contains a realization of each strong type of p. By 1.1 we can require that there is no more than one realization of each strong type in A.

§2. The coordinate tree

Throughout this section we suppose T has few models and that M is an arbitrary uncountable model of T.

First some terminology about trees. For a tree $(\tau, <)$ and $u \in \tau$ we write u^- for the predecessor of u, i.e., the $v \in \tau$ such that $w < u \Rightarrow w < v$ or $w = v$. We write u^+ for $\{v \in \tau : v^- = u\}$, $u_> = \{v \in \tau : u < v\}$. If $u^+ = \emptyset$ we call u a *leaf*. $s \subset \tau$ is called an *ideal* if $v \in s$ and $w < v \Rightarrow w \in s$. $u_< = \{v \in \tau : v < u\}$, $u_\leqq = u_< \cup \{u\}$. All trees will have $\emptyset$ as the least element. u is said to be on the *i-th level* if $|u_<| = i$. For all our trees τ there is a $k \in \omega$ such that $|u_<| \leqq k$ for all $u \in \tau$. The smallest such k is called the *height* of τ.

Definition 2.1. We define a coordinate tree of M one level at a time. Let $L_0 = \{\emptyset\}$. Suppose L_i has been defined and $u \in L_i$. Let $A_0(u), \dots, A_n(u) \subset M^{eq}$ be a set of $u_\leqq$-definable reduced rank 1 atoms satisfying

(a) if $i \neq j$ $A_i(u) \perp A_j(u)$;

(b) for all $i \leqq n$ $A_i(u) \dashv u_<$, except when $u = \emptyset$;

(c) if $C \subset M^{eq}$ has rank 1, $C \not\dashv u_\leqq$ and $C \dashv u_<$, then $C \not\perp A_i(u)$ for some $i \leqq n$.

(d) We further suppose that whenever $v \in L_i$ is such that $v_\leqq \equiv u_<$, $A_i(u)$ is conjugate to $A_i(v)$ for all $i \leqq n$.

Let $B_i(u)$ be a basis for $A_i(u)$ in M^{eq}. Let $L_{i+1} = \bigcup \{B_i(u) : i \leqq n,\ u \in L_i\}$. Extend the order $<$ by: $v \in B_0(u) \cup \cdots \cup B_n(u)$ iff $u < v$, for all $v \in L_{i+1}$.

Let j be the least number such that $L_{j+1} = \emptyset$. Let $\tau = L_0 \cup \cdots \cup L_j$ and call $(\tau, <)$ a *coordinate tree of* M. We say a type p *appears* in τ if there is a $u \in \tau$ such that $p \in S(u_<)$ and p is realized in $A_i(u)$ for some $i \leqq n$. For $i \leqq n$ let $\tau_i = (L_0 \cup \cdots \cup L_i, <)$.

Remark 2.2. Since not all $u_\leqq$ above are conjugate the n does depend on u. To simplify notation we chose not to express this dependence.

Lemma 2.3. *M has a coordinate tree.*

Proof. We must show it's possible to carry out the above construction. Suppose we can define τ_i and $u \in L_i$. By 1.4 and 1.7 there is a finite set $A_0(u), \dots, A_n(u)$ satisfying (a)–(c). To satisfy (d) notice that if $v_< \equiv u_<$ then the conjugates over $u_\leqq$ of $A_0(v), \dots, A_n(v)$ satisfy (a)–(c) for u. That there is a j such that $L_{j+1} = \emptyset$ follows from the finiteness of $\mathrm{rk}(M)$ and

Claim 2.4. *If $\emptyset < v < u$ then $u \not\perp_{v_<} v$ and u is dominated by v over $v_<$.*

Recall the domination relation on sets [10, C.10]. With u, v as above let

$s = \{w : v < w \leqslant u\}$. Note that $w \in s \Rightarrow \mathrm{t}(w/w_<) \dashv v_<$. Using this it is an easy exercise to show that s is dominated by v over $v_<$. In particular, u is dominated by v over $v_<$. The claim is now clear.

From here on τ denotes this coordinate tree of M. Recall that $\mathrm{dcl}(A) = \{a : \text{there is a formula } \psi(x) \text{ over } A \text{ such that } \models \psi(a) \wedge \exists! x \psi(x)\}$.

COROLLARY 2.5. $u \in \tau \Rightarrow u_< \in \mathrm{dcl}(u)$.

PROOF. Since for any A $\mathrm{dcl}(\mathrm{dcl}(A)) = \mathrm{dcl}(A)$ it suffices to show that for all $v < u$

$$v \in \mathrm{dcl}(uv_<). \tag{1}$$

By 2.4 $(v \not\downarrow u/v_<)$ and u is dominated by v over $v_<$. Let $\varphi = \mathrm{t}(v/v_<)$. Thus, $v \in \mathrm{acl}(uv_<) \cap \varphi(M)$ and $w \in \mathrm{acl}(uv_<) \cap \varphi(M) \Rightarrow (w \not\downarrow v/v_<)$. Thus, $\mathrm{acl}(uv_<) \cap \varphi(M) = \mathrm{acl}(v_\leqq) \cap \varphi(M) = \{v\}$, proving (1).

It is clear that sets definable over $\mathrm{dcl}(A)$ are also definable over A. Thus, for $u \in \tau$, $\mathrm{t}(u/u_<)$ is the unique extension of $\mathrm{t}(u/u^-)$. So, from here on we will replace $u_\leqq$ by u in most contexts.

LEMMA 2.6. *There are finitely many conjugacy classes of types appearing in* τ.

PROOF. We prove by induction that there are finitely many conjugacy classes of types appearing in τ_i. Suppose it's true for τ_i. If $u, v \in I_i$ and $u \equiv v$ then 2.1(d) implies that the types over u realized in u^+ are conjugate to the types over v realized in v^+. It now follows from the inductive hypothesis that τ_{i+1} has the desired property.

From here on M^* denotes an extension by definitions which contains τ. Its existence is guaranteed by 2.6.

LEMMA 2.7. τ *is independent with respect to* $<$.

PROOF. We prove by induction that for all i, τ_i is independent with respect to $<$. For τ_0 this is trivial and for τ_1 it follows from 2.1(a). Now suppose τ_i is independent and $u \in L_i$, for $i > 1$. As in the τ_1 case we have

$$u^+ \text{ is } u\text{-independent}. \tag{2}$$

Every $p \in S(u)$ realized in u^+ is orthogonal to u^-, so by (2) $t(u^+/u) \dashv u^-$. Let $s = \tau_i \setminus \{u\}$. By the independence of τ_i $(u \downarrow s/u^-)$. Hence, by [10, C.8] $t(u^+/u) \dashv s$, yielding

$$u^+ \cup \{u\} \downarrow_{u^-} s \cup \bigcup \{v^+ : v \in L_i \cap s\}. \tag{3}$$

From (3) it's clear that the set of leaves of τ_{i+1} are τ_i-independent. The independence of τ_{i+1} follows easily from this.

DEFINITION 2.8. For any tree σ let $\sigma' = \sigma \setminus \{v : v \text{ is a leaf}\}$.

LEMMA 2.9. *$u \in \tau' \Rightarrow t(u/u^-)$ is trivial.*

PROOF. Suppose $v \in u^+$. By 2.1(b) $t(v/u) \dashv u^-$. Now the lemma follows immediately from 1.8(i) and our assumption that T has few models.

LEMMA 2.10. *If p appears in τ' then $p(M^*) \subset \tau'$.*

PROOF. Suppose $p \in S(u)$ and $B = \tau' \cap p(M^*)$. We chose B to be a basis for p in M^{eq}, thus $c \in p(M^*) \Rightarrow c \not\downarrow_u B$. The fact that p is trivial and reduced implies immediately that $c \in B$, proving the lemma.

Some of our freedom in the choice of a coordinate tree is due to our choice of representatives from non-orthogonality classes (see 2.1(c)). We will see this is almost the only point of freedom. Fix M^* and τ a coordinate tree of M. Let $\sigma \subset \mathfrak{C}^{eq}$ be an independent tree. We call σ *good* if every type appearing in σ is conjugate to a type appearing in τ, and conversely.

LEMMA 2.11. *Every model N has a good coordinate tree.*

PROOF. This is proved by induction, as usual. Suppose we have found σ_i, the first i levels in a coordinate tree for N, such that every type appearing in σ_i is conjugate to a type appearing in τ_i, and conversely. Then it is easy to show $u \in \sigma_i$ implies $u_{\leq} \equiv v_{\leq}$ for some $v \in \tau_i$. So, if $A_0(v), \ldots, A_n(v)$ are the $v_{\leq}$-definable types used to define v^+, their conjugates may be used to define u^+ (since the conditions (a)–(c) also hold for these conjugates). The lemma now follows easily.

LEMMA 2.12. *If σ is another coordinate tree for M which is good, then $\sigma' = \tau'$.*

PROOF. Let $\sigma'_i = \sigma_i \cap \sigma'$, similarly define τ'_i. Suppose $\sigma'_i = \tau'_i$ and let $u \in L_i$ not be a leaf of τ'. Let $A_0, \ldots, A_n$ be the u-definable sets appearing in τ. Let $B_0, \ldots, B_m$ be the u-definable sets appearing in σ. Since σ is good there is a $w \in \tau$ and w-definable sets $C_0, \ldots, C_m$ which appear in τ and are conjugate to $B_0, \ldots, B_m$. These sets satisfy 2.1(a)–(c) for w, so by 2.1(d) we know each C_i is conjugate to some A_j. Thus, renumbering if necessary, we have $m = n$ and $B_i = A_i$ for $i \leqq n$.

We are not finished with the proof since some of the A_i's may not appear in τ'. Note that A_i appears in τ' iff (*) there is a $v \in A_i$ and a v-definable rank 1 set D such that $D \dashv v^-$. Notice (*) also guarantees A_i appears in σ'. Thus, we can number the A_i's so that $A_0, \ldots, A_j$ are the sets appearing in both τ' and σ'. By 2.10 $u^+ \cap \tau' = A_0 \cup \cdots \cup A_j = \sigma' \cap \{v \in \sigma\colon\ v^- = u\}$. This proves $\sigma'_{i+1} = \tau'_{i+1}$ yielding the lemma.

REMARK 2.13. Not only do we have $\sigma' = \tau'$, but as we proved in the first paragraph above, the same types appear in $\sigma \backslash \sigma'$ and $\tau \backslash \tau'$. Thus, σ differs from τ only in a choice of basis for these sets realized by leaves.

An easy induction argument using 2.5, 2.10 and the fact that there are finitely many conjugacy classes of types appearing in τ' shows that τ' is definable. If we were willing to sacrifice the independence of the set of leaves we could alter 2.1 by adding all of $A_i(u)$ instead of just a basis. The result is a definable tree which serves our purpose equally as well. We chose the present definition only to make the above proofs easier. By these remarks we have

PROPOSITION 2.14. *If σ and π are good coordinate trees for N, then $\sigma \cong \pi$ (as trees).*

Our long-range goal is to show M is prime over τ. In Shelah's treatment, where the tree is a collection of submodels, this is not so difficult. The fact that the base of the tree is a model allows the application of strong results about domination and non-orthogonality to a set (see [5, 4.1, 4.3]). In our context there is more work involved.

PROPOSITION 2.15. *Suppose p in $S(M^{eq})$ has rank 1 and is non-orthogonal to τ. Then there is a q appearing in τ such that $p \not\perp q$.*

PROOF. Let $s \subset \tau$ be a minimal ideal such that $p \not\dashv s$.

CLAIM. *There is a $u \in \tau$ such that $s = u_{\leqq}$.*

Assume, towards a contradiction, that s contains at least two $<$-maximal elements v, v'. Let $s' = s \backslash \{v, v'\}$. By the minimality assumption

$$p \dashv s'v \quad \text{and} \quad p \dashv s'v'. \tag{4}$$

Since s is an ideal both v and v' have rank 1 over s'. If $v \not\perp_{s'} v'$ then $v' \in \mathrm{acl}(s'v)$, implying $p \not\dashv s'v$ to contradict (4). Thus, $v \downarrow_{s'} v'$, which combined with (4) contradicts 1.9, proving the claim.

By the minimality assumption on $s = u_{\leqq}$, $p \dashv u^-$. By 2.1(c) there is a $q \in S(u)$ appearing in τ such that $q \not\perp p$. This proves the proposition.

LEMMA 2.16. *Suppose B is finite, $A \subset M^{eq}$ is B-definable, strongly minimal and $A \not\dashv \tau$. Then*

(a) *for all but finitely many* $b \in A$, $A \subset \mathrm{acl}(\tau Bb)$,
(b) *for all sets* $C \subset M^*$, *A is dominated by* $\tau \cup B \cup C$ *over* $\emptyset$,
(c) *if A is modular*, $A \subset \mathrm{acl}(\tau B)$.

PROOF. By 2.15 there is a rank 1 set D appearing in τ such that $D \not\perp A$. Suppose D is definable over w. Let $b \in A \setminus \mathrm{acl}(Bw)$. By Lemma 1.1(i), $A \subset \mathrm{acl}(bBwD)$. A fortiori, $A \subset \mathrm{acl}(bB\tau)$, to prove (a). For (c) note that by 1.1(ii), $A \subset \mathrm{acl}(BwD)$.

Turning to (b) suppose $d \in \mathfrak{C}^{eq}$ is such that $d \downarrow \tau BC$. Clearly, it suffices to prove that $d \downarrow \tau BCA$ under the assumption that C is finite. Let w be as in the proof of (a), $b \in A \setminus \mathrm{acl}(BCwd)$. As above $A \subset \mathrm{acl}(B\tau b)$. Since $b \downarrow_{CBw} d$ and $d \downarrow CBw$, $d \downarrow \tau BCA$, as desired.

Recall that for a sequence $\{e_i : i < \alpha\}$ we abbreviate $\{e_i : i < \beta\}$ by E_β.

DEFINITION 2.17. Let A be any set, $a \in \mathfrak{C}^{eq}$. We call $\{(c_i, A_i) : i \leqq n\}$ a *filtration of a over A* if
(a) each A_i is a reduced rank 1 atom which is definable over $A \cup C_i$,
(b) $c_i = \mathrm{acl}(AC_ia) \cap A_i \neq \emptyset$,
(c) $a \in \mathrm{acl}(AC_{n+1})$.

REMARK 2.18. The purpose of a filtration is to "construct" a in terms of elements of rank 1. A filtration accomplishes this in the following sense. For simplicity let $A = \emptyset$. Each element of c_i is in A_i, hence has rank 1 over C_i by (a). Let $E = c_0 \cup \cdots \cup c_n = \{e_i : i \leqq k\}$, ordered so that each element of c_j comes before each element of c_{j+1} ($j < n$). (Remember that the c_j's are not elements, they are finite subsets of the A_j's.) Thus, E gives us a sequence such that $\mathrm{rk}(e_j / E_j) \leqq 1$ and $a \in \mathrm{acl}(E)$.

LEMMA 2.19. *For all finite A and a there is an extension by definitions* $\mathfrak{C}^*$ *in which there is a filtration of a over A.*

PROOF. This is proved with the Coordinatization Theorem. By 1.2 there is a reduced rank 1 A-definable atom A_0 such that $c_0 = \mathrm{acl}(Aa) \cap A_0 \neq \emptyset$. Now suppose A_{i-1} and c_{i-1} have been defined so that $\mathrm{t}(a / AC_i)$ is non-algebraic. Let A_i be a reduced rank 1 atom definable over $A \cup C_i$ such that $c_i = \mathrm{acl}(AC_ia) \cap A_i \neq \emptyset$. By ω-stability and the fact that $(a \not\downarrow c_i / AC_i)$ there is an $n < \omega$ such that $\mathrm{t}(a / AC_{n+1})$ is algebraic. $\{(c_i, A_i) : i \leqq n\}$ is a filtration of a over A.

DEFINITION 2.20. T is said to be of *modular type* if for all a and finite A in $\mathfrak{C}^{eq}$

there is a filtration $\{(c_i, A_i): i \leqq n\}$ of a over A such that every strong type of A_i is modular, for $i \leqq n$.

PROPOSITION 2.21. *Let $C \subset M^*$ and $a \in M^*$. Then* $\mathrm{t}(a/\tau C)$ *is atomic. Furthermore, a is dominated by $\tau \cup C$ over $\emptyset$.*

PROOF. The basic idea behind the proof is to take a filtration of a over τ and "replace" the A_i's by sets appearing in τ using 2.16. The hard part is showing each A_i is non-orthogonal to τ.

Let $\{(c_i, A_i): i \leqq n\}$ be a filtration of a over D, where $D \subset \tau$ is a finite set such that $a \downarrow_D \tau$. (If $\mathrm{t}(a/\tau)$ is algebraic there is nothing to prove.)

CLAIM 2.22. *For each $i \leqq n$ there is a b_i such that*
(a) $b_i \subset A_i$,
(b) $A_i \subset \mathrm{acl}(\tau B_{i+1})$,
(c) $\mathrm{t}(b_i/\tau C B_i)$ *is atomic,*
(d) *if each strong type of A_i is modular,* $b_i \in \mathrm{acl}(\tau B_i)$.

We choose the b_j's by recursion. Let $D_0, \ldots, D_k$ be the strongly minimal components of A_0. By 2.16, for all but finitely many $e_i \in D_i$, $D_i \subset \mathrm{acl}(\tau e_i)$. Thus, we can choose these e_i's so that the type of $b_0 = e_0 \cdots e_k$ over τC is atomic, giving (a)–(d) for $i = 0$. Suppose $b_0, \ldots, b_{j-1}$ have been chosen so that (a)–(d) hold. To find b_j we must first prove

SUBCLAIM 2.23. $A_j \not\perp \tau$.

Suppose not. By (b) and 2.17(a) we have

(5) $$\text{for } l \leqq j,\ A_l \text{ is definable over } \mathrm{acl}(\tau B_l).$$

So $A_j \not\perp \tau B_j$. Let $k \leqq j$ be such that $A_j \not\perp \tau B_k$ and $A_j \dashv \tau B_{k-1}$. Let $e \subset b_{k-1}$ be minimal so that

(6) $$A_j \not\perp \tau B_{k-1} e \quad \text{and} \quad A_j \dashv \tau B_{k-1}.$$

First suppose e is a singleton. Then by (5), $\mathrm{rk}(e/\tau B_{k-1}) = 1$. By 2.22(d) we know $\mathrm{t}(e/\tau B_{k-1})$ is non-modular (otherwise $A_j \not\perp \tau B_{k-1}$). Combining these facts we contradict 1.8(i). If e is not a singleton it is a τB_{k-1}-independent set of elements of rank 1. We easily get a contradiction to 1.9(ii), to prove 2.23.

Since A_j is an atom each of its strong types is non-orthogonal to τ. Apply 2.16 to each strong type of A_j to obtain a sequence $b_j \subset A_j$ to satisfy (a)–(c). Part (d) of the claim follows from 2.16(c).

Combining the definition of a filtration with 2.22(b) we see that $a \in \mathrm{acl}(\tau B_{n+1})$.

By 2.22(c), B_{n+1} is atomic over τC. By the transitivity of isolation we have $t(a/\tau C)$ isolated. Notice that if T is of modular type $a \in \mathrm{acl}(\tau)$. To finish the proof of 2.21 it suffices to prove

CLAIM 2.24. *a is dominated by $\tau \cup C$ over $\varnothing$.*

By 2.23, 2.16(b) and the fact that A_{i+1} is definable over $\tau \cup A_0 \cup \cdots \cup A_i$, we have A_{i+1} dominated by $\tau \cup C \cup A_0 \cup \cdots \cup A_i$ over $\varnothing$. By the transitivity of domination $A_0 \cup \cdots \cup A_i$ is dominated by τC over $\varnothing$. Since $a \in \mathrm{acl}(\tau A_0 \cdots A_n)$ 2.24 is proved, as well as 2.21.

THEOREM 1. *Suppose T has few models, $M \models T$ and τ is a coordinate tree of M. Then M is prime over τ and dominated by τ over $\varnothing$. If T is of modular type then $M \subset \mathrm{acl}(\tau)$.*

PROOF. The last sentence follows immediately from 2.22. Let $N \subset M$ be a maximal atomic model over τ and fix a construction $\langle c_\alpha : \alpha < |N| \rangle$ of N. By 2.21, $N = M$. To prove the domination notice that by 2.21 at stage α of the construction we have C_α dominated by τ over $\varnothing$, and c_α dominated by τC_α over $\varnothing$. By the transitivity of domination $C_{\alpha+1}$ is dominated by τ over $\varnothing$. The theorem follows easily.

The last sentence of Theorem 1 generalizes the following theorem of Gisela Ahlbrandt [1]: *if T is totally categorical of modular type, then T is almost strongly minimal.*

LEMMA 2.25. *Suppose $A \subset M^{\mathrm{eq}}$ and $p \in S_1(A)$ is strongly minimal. Then $p \not\perp \tau$, hence p is non-orthogonal to some type appearing in τ.*

PROOF. Wlog, $A = a$ is finite. 2.22 yields a sequence $E = \{e_i : i \leqq k\}$ such that $a \in \mathrm{acl}(\tau E)$ and for all $i \leqq k$, $t(e_i/\tau E_i)$ has rank 1 and is non-orthogonal to τ. p is a type over $\mathrm{acl}(\tau E)$. The proof that p is non-orthogonal to τ is exactly like the proof of 2.23.

We now connect our results with the concept of *depth* found in [5]. Assuming T has few models we know it is shallow. It's not hard to see the depth of T must be finite.

PROPOSITION 2.26. *The depth of T is the height of any good coordinate tree.*

PROOF. We use some concepts and results from [5, §6]. The definitions imply $d(T) = \max\{d(p)+1 : p \text{ a stationary type}\}$. By [5, 6.2(ii)] we can restrict these p to regular types. It's an easy exercise (using 1.2) to show every regular type is non-orthogonal to a strongly minimal type in $\mathfrak{C}^{eq}$. Let σ be a coordinate tree for $\mathfrak{C}$. By 2.25 every strongly minimal type in $\mathfrak{C}^{eq}$ is non-orthogonal to a type appearing in σ. Thus, $d(T) = \max\{d(p)+1\colon p$ a strong type of a type appearing in $\sigma\}$. Let n be the height of σ.

CLAIM. *If p is a strong type of a type appearing in σ and realized in the i-th level, $d(p) \leqq n - i$.*

The claim is proved by induction on $n - i$. Suppose $n - i = 0$ and $p = \mathrm{stp}(u/u_<)$. Let $N \models T^{eq}$ be a countable model containing $u_<$ such that $u \downarrow_{u_<} N$ and let $M = N[u]$. Let $q \in S(M)$ be strongly minimal. To show that $d(p) = 0$ it suffices to prove that $q \not\perp N$. By Lemma 2.25 there is a $v \in \sigma$ such that $q \not\perp \mathrm{stp}(v/v_<)$. Let $A = M \cap v_<$.

SUBCLAIM. $A = v_<$.

Suppose $v_< \backslash A$ is non-empty and enumerate it as $w_0, \ldots, w_k$ with $w_{i+1}^{-} = w_i$ $(i < k)$. (This is possible by 2.5 and the fact that $\mathrm{dcl}(M) = M$.) Since $\mathrm{rk}(w_0/A) = 1$, $M \not\downarrow_A w_0$ implies $w_0 \in \mathrm{acl}(M) = M$. Thus $M \downarrow_A w_0$. By Claim 2.4, w_k is dominated by w_0 over A, so $w_k \downarrow_A M$. Since $(w_k)_< \subset \mathrm{dcl}(w_k)$ we have $v_< \downarrow_A M$. Our assumption that $v_< \neq A$ implies that $\mathrm{t}(v/v_<) \dashv A$ (by 2.1(b)). By [10, C.8] $\mathrm{t}(v/v_<) \dashv M$. This contradicts that $q \in S(M)$ and $q \not\perp \mathrm{t}(v/v_<)$, proving the subclaim.

By the subclaim we have $v_< \in M$. Since $a \in M \backslash N \Rightarrow a \not\downarrow_N u$ it is easy to show that any $w \in \sigma \cap (M \backslash N)$ is also a leaf. Thus, $v_< \in N$. Hence $\mathrm{t}(v/v_<)$ and q are both non-orthogonal to N.

The general inductive step is organized as follows. Suppose $p = \mathrm{stp}(u/u_<)$, u is on the i-th level, and M and N are chosen as above. Let $q \in S(M)$ be strongly minimal, $v \in \sigma$ such that $\mathrm{t}(v/v_<) \not\perp q$. As in the proof of the subclaim we have $v_< \in M$. Since $a \in M \backslash N \Rightarrow a \not\downarrow_N u$, we see that $v_< \in L_0 \cup \cdots \cup L_i$. Thus, v is on at most the $i+1$-st level. It follows from the inductive hypothesis that $d(q) \leqq n - (i+1)$. By the definition of depth $d(p) \leqq n - i$. This proves the claim.

Now suppose $u \in L_1$ is such that for all i, $2 \leqq i \leqq n$, $u_> \cap L_i \neq \emptyset$, $p = \mathrm{stp}(u)$. Clearly, $d(p) \geqq n - 1$. By the claim $d(p) = n - 1$ and this is the maximum depth among such types. This proves the proposition.

§3. The spectrum function and invariants for M

We continue the conventions adopted in Section 2: M is an arbitrary uncountable model, τ is a coordinate tree for M and all coordinate trees are good. We showed in Section 2 that there is a one-one correspondence between models and isomorphism types of coordinate trees. In this section we first give a necessary and sufficient condition for a tree to be the coordinate tree of some model. The spectrum function is then determined by counting the number of these trees. Lastly, we give a method of labelling coordinate trees which generates an invariant for M.

DEFINITION 3.1. We say a good tree σ is *permissible* if the following hold whenever $u \in \sigma$ and $p \in S(u)$ is conjugate to a type appearing in some coordinate tree.

(a) Suppose the strong types of p are modular. Then for all strong types q of p, $q(\sigma)$ is infinite.

(b) Suppose the strong types of p are non-modular. Then there are strong types $q_0, \ldots, q_m$ of p which represent the non-orthogonality classes among the strong types of p such that $q_i(\sigma)$ is infinite for $i \leqq m$. σ also contains a minimal set of realizations of other strong types of p such that each strong type of p is realized in $\mathrm{acl}(p(\sigma) \cup u)$.

PROPOSITION 3.2. *If σ is permissible there is an N such that σ is a coordinate tree for N.*

PROOF. Let N be prime over σ. Our definition of permissible guarantees that for $u \in \sigma$ there are $A_0(u), \ldots, A_n(u)$ appearing in σ which satisfy 2.1(a)–(c). Notice also that 2.1(d) is satisfied. So, to show σ is a coordinate tree it suffices to show that for $p \in S(u)$ appearing in σ, $B = p(\sigma)$ is a basis for p in N. (See our remarks on bases of reduced rank 1 atoms in Section 1.) Let $c \in p(N)$. Using the independence of σ and the fact that $q \in S(v)$ appearing in σ implies $q \dashv v^{-}$, it is routine (but tedious) to show $(c \downarrow \sigma / Bu)$. Thus, $\mathrm{t}(c/Bu)$ is atomic. Let $D \subset Bu$ be finite and such that

$$(1) \qquad \mathrm{t}(c/D) \vdash \mathrm{t}(c/Bu) \quad \text{and} \quad u \in D.$$

We may assume D contains a realization of all strong types q of p which are realized in B. Since B is infinite there is a $b \in B \backslash D$. If $\mathrm{t}(c/D)$ is non-algebraic 3.1 implies $\mathrm{t}(b/D) \not\perp^{w} \mathrm{t}(c/D)$. This contradicts (1). Thus, $\mathrm{t}(c/D)$ is algebraic to prove the proposition.

We now show how to count the number of non-isomorphic permissible trees of cardinality $\aleph_\alpha$. Let d be the height of any coordinate tree (d is the depth of T by 2.26). One case is a particular nuisance so we consider it separately.

Subsection 3.3. (The spectrum function when $d = 1$) Wlog, T is not $\aleph_1$-categorical. Let $q_0, \dots, q_{n-1}$ be the strong types over $\varnothing$ which are realized by infinitely many elements of τ. Let $e_i \in \operatorname{acl}(\varnothing)$ be such that there is a stationary $r_i \in S(e_i)$ parallel to q_i. Let $\bar{r} = \bigcup_{i<n} r_i(x_i)$, which is easily seen to be in $S_n(E_n)$ since the q_i's are pairwise orthogonal.

This case is complicated by the fact that there may be a structure placed on the e_i's which keeps permutations of the r_i's from being conjugate to $\bar{r}$.

Let $\Lambda_\alpha = \{\kappa : \aleph_0 \leqq \kappa \leqq \aleph_\alpha\}$, Λ_α^n the set of functions from n into Λ_α. We define an equivalence relation $\sim$ on Λ_α^n (uniformly for all α) as follows:

$f \sim g$ if there is a permutation π of n such that $f = g \circ \pi$ and $\bigcup_{i<n} r_{\pi(i)}(x_{\pi(i)})$ is conjugate to $\bar{r}$.

Let $f_M \in \Lambda_\alpha^n$ be such that $f_M(i) = \dim(r_i(M))$ for $i < n$. We leave it as an exercise to show $f_M/\sim$ is an invariant for M, i.e.

$$(2) \qquad M \cong N \quad \text{iff} \quad f_M/\sim \; = f_N/\sim .$$

Let $I_*(\lambda, T)$ be the number of non-isomorphic models of cardinality at most λ. We have

$$(3) \qquad I_*(\aleph_\alpha, T) = |\Lambda_\alpha^n/\sim| \quad \text{and} \quad I(\aleph_\alpha, T) = \Big| (\Lambda_\alpha^n/\sim) \setminus \bigcup_{\beta<\alpha} (\Lambda_\beta^n/\sim) \Big| .$$

First suppose α is infinite. Since each $\sim$-class is finite and Λ_α^n is infinite

$$(4) \; I_*(\aleph_\alpha, T) = |\Lambda_\alpha^n| = |\alpha + 1| \quad \text{and} \quad I(\aleph_\alpha, T) = \Big| \Lambda_\alpha^n \setminus \bigcup_{\beta<\alpha} (\Lambda_\beta^n) \Big|, \qquad \text{when } \alpha \geqq \omega.$$

It is easy to see $|\Lambda_\alpha^n \setminus \bigcup_{\beta<\alpha} (\Lambda_\beta^n)| = |\Lambda_\alpha^n|$, so

$$(5) \qquad I(\aleph_\alpha, T) = |\alpha|, \qquad \text{when } \alpha \geqq \omega.$$

For α finite we have

$$(6) \qquad I(\aleph_{\alpha+1}, T) = |(\Lambda_{\alpha+1}^n/\sim) \setminus (\Lambda_\alpha^n/\sim)| = |\Lambda_{\alpha+1}^n/\sim| - |\Lambda_\alpha^n/\sim|.$$

To get an idea of what this number is notice that $n \cdot |\alpha + 1| \leqq |\Lambda_\alpha^n/\sim| \leqq |\alpha + 1|^n$. Note that for a given theory $\sim$ is uniquely determined, so $I(\lambda, T)$ can be computed precisely.

REMARK 3.4. It follows from the next proposition and the theorem quoted in the title of [6] that 3.3 gives the only possible spectra for theories with a finite number of models in some uncountable power.

PROPOSITION 3.5. *If $d \geqq 2$ then for $\alpha > 0$*

$$I(\aleph_\alpha, T) = \beth_{d-2}((|\alpha| + \aleph_0)^{|\alpha|}. \tag{7}$$

PROOF. Starting with elements on the highest level and working down we count the possible isomorphisms types over u for $u_>$. Suppose $u \in L_{d-1}$ and $u^+ \neq \emptyset$. As in the $d = 1$ case we find that the number of isomorphism types for $u_>$ is

$$\text{finite if } \alpha < \omega \quad \text{and} \quad |\alpha| \text{ if } \alpha \geqq \omega. \tag{8}$$

Now suppose $u \in L_{d-2}$ and $u_> \cap L_d \neq \emptyset$. Let q be a strong type over u realized by infinitely many $v \in u^+$ such that $v^+ \neq \emptyset$. (Since q is trivial this property is independent of the choice of permissible tree.) For a fixed coordinate tree σ containing u the isomorphism type over u of $(q,\sigma)_> = q(\sigma) \cup \{w : w^- \in q(\sigma)\}$ is determined by the number of elements v of $q(\sigma)$ such that v^+ has a fixed isomorphism type. Note that there are permissible trees where this number is finite. Combining this remark with (8) we find that the number of isomorphism types in $\{(q,\sigma)_> : u \in \sigma, \sigma \text{ a coordinate tree}\}$ is $(|\alpha| + \aleph_0)^{|\alpha|}$. Letting q range over the strong types realized in u^+ we see that this is also the number of isomorphism types for $u_>$. For $u \in L_{d-3}$ with $u_> \cap L_d \neq \emptyset$ we similarly find that the number of isomorphism types for $u_>$ is $\beth_1((|\alpha| + \aleph_0)^{|\alpha|})$. Continuing down through the levels we eventually compute $\emptyset_>$, the number of non-isomorphic coordinate trees, as $\beth_{d-2}((|\alpha| + \aleph_0)^{|\alpha|})$, to prove the theorem.

We now show how to label a coordinate tree of M to find a set-theoretically simple invariant which determines M up to isomorphism. We first need to discuss a problem like that which arose in the $d = 1$ case. Let σ be a coordinate tree of N, $u \in \sigma'$. Let $r_0, \ldots, r_{n-1}$ be strong types over u realized by infinitely many elements of u^+. As before we may assume there is an $e_i \in \mathrm{acl}(u)$ such that $r_i \in S(e_i u)$. Let $\bar{r}_u = \bigcup_{i<n} r_i(x_i)$ (an element of $S_n(E_n u)$). Recall that τ denotes the coordinate tree of M obtained in Section 2. We may require that whenever $u, v \in \tau'$ are conjugate, $\bar{r}_u$ and $\bar{r}_v$ are conjugate. We call a good coordinate tree σ *excellent* if whenever $v \in \sigma$ is conjugate to $u \in \tau$, $\bar{r}_v$ is conjugate to $\bar{r}_u$. As in 2.11 every model has an excellent coordinate tree.

Now we're ready to label τ. Let $|M| = \lambda$ and $\Lambda = \{\kappa : \kappa \leqq \lambda\}$ (including finite cardinals). We will associate with $u \in \tau$ a set tag(u) which determines $u_>$ up to isomorphism. The labelling is done from the top down. If u is a leaf let tag$(u) = \{\emptyset\}$. Suppose tag(v) has been defined for all elements on levels above i and $u \in L_i$. Let $F_{i+1} = \{\text{tag}(v) : v \in L_{i+1}\}$. Let $r_0, \ldots, r_{n-1}$ and $\bar{r}_u$ be as above. For

$j < n$ let $f_j : F_{i+1} \to \Lambda$ be given by

(9) for $\xi \in F_{i+1}$, $f_j(\xi) = |\{v \in u^+ \colon v \text{ realizes } r_j \text{ and } \mathrm{tag}(v) = \xi\}|$.

Finally let $\mathrm{tag}(u) = \{f_{\pi(0)}, \ldots, f_{\pi(n-1)}\rangle$: π is a permutation of n such that $\bigcup_{i<n} r_{\pi(i)}(x_{\pi(i)})$ is conjugate to $\bar{r}_u\}$.

We define the *invariant* of M, $\mathcal{I}(M)$, to be $\mathrm{tag}(\emptyset)$. Assuming that, for all N, $\mathcal{I}(N)$ is computed with an excellent coordinate tree we easily prove

THEOREM 3.6. *$M \cong N$ iff $\mathcal{I}(M) = \mathcal{I}(N)$.*

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